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2009 J. Phys. A: Math. Theor. 42 322001

(http://iopscience.iop.org/1751-8121/42/32/322001)

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IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 322001 (6pp) doi:10.1088/1751-8113/42/32/322001

FAST TRACK COMMUNICATION

On spacetime rotation invariance, spin-charge separation and $SU(2)$ Yang–Mills theory

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Received 14 March 2009, in final form 8 June 2009
Published 21 July 2009
Online at stacks.iop.org/JPhysA/42/322001

Abstract
Previously, it has been shown that in a spin-charge separated $SU(2)$ Yang–Mills theory, (Euclidean) spacetime rotation invariance can be broken by an infinitesimal 1-cocycle that appears in the $SO(4)$ boosts. Here we study in detail the structure of this 1-cocycle. In particular, we show that its non-triviality relates to the presence of a (Dirac) magnetic monopole bundle. We also compute the finite version of the cocycle.

PACS numbers: 11.15.Tk 02.40.Hw, 02.40.Re

1. Introduction

Recently, the properties of four-dimensional Euclidean $SU(2)$ Yang–Mills theory have been investigated using spin-charge separated variables [1, 2]. Such variables might be valuable in describing the confining strong coupling regime of the theory [3]. In [1], it was shown that even though these variables reveal the presence of two massless Goldstone modes, this apparent contradiction with the existence of a mass gap becomes resolved since these Goldstone modes break spacetime rotation invariance by a 1-cocycle [1]: since the ground state must be spacetime rotation invariant the 1-cocycle is to be removed. This demand fixes the ground state uniquely and deletes all massless states from the spectrum [1].

In [1] only the infinitesimal form of the 1-cocycle was presented. Here we display its finite form. We also verify that the 1-cocycle is indeed non-trivial, by relating it to the nontriviality of the Dirac magnetic monopole bundle.

2. Spin-charge separation

We consider the Lagrangian of $SU(2)$ Yang–Mills theory, with the maximal Abelian gauge condition [1]. This eliminates the gauge freedom in the $SU(2)/U(1)$ submanifold of the gauge group, leaving us with a $U(1)$ gauge symmetry; see [1] for a detailed discussion including gauge covariance. The spin-charge separation then amounts to the following decomposition of the off-diagonal components $A^{\pm}_{\mu}$ of the gauge field A^{a}_{μ} [1, 3]

$$A^{+}_{\mu} = A^{1}_{\mu} + \mathrm{i}A^{2}_{\mu} = \psi_1 e_\mu + \psi_2 e^{\star}_{\mu}. \tag{1}$$

Here the spin field e_μ

$$e_\mu = \tfrac{1}{\sqrt{2}}\left(e^{1}_{\mu} + \mathrm{i}e^{2}_{\mu}\right)$$

is normalized according to

$$e_\mu e_\mu = 0, \qquad e_\mu e^{\star}_{\mu} = 1. \tag{2}$$

This can be viewed as a Clebsch–Gordan type decomposition of $A^{\pm}_{\mu}$, when interpreted as a tensor product of the complex spin variable e_μ that remains intact under $SU(2)$ gauge transformations and the charge variables $\psi_{1,2}$ that are spacetime scalars but transform nontrivially under $SU(2)$ [1].

The decomposition introduces an internal $U_I(1)\times\mathbb{Z}_2$ symmetry that is not visible to A^{a}_{μ}. The $U_I(1)$ action is

$$U_I(1):\quad \begin{array}{ccc} e_\mu & \rightarrow & \mathrm{e}^{-\mathrm{i}\lambda}e_\mu, \\ \psi_1 & \rightarrow & \mathrm{e}^{\mathrm{i}\lambda}\psi_1, \\ \psi_2 & \rightarrow & \mathrm{e}^{-\mathrm{i}\lambda}\psi_2. \end{array} \tag{3}$$

This is a local frame rotation, in particular it preserves the orientation in e_μ. The $\mathbb{Z}_2$ action exchanges ψ_1 and ψ_2,

$$\mathbb{Z}_2:\quad \begin{array}{ccc} e_\mu & \rightarrow & e^{\star}_{\mu} \\ \psi_1 & \rightarrow & \psi_2 \\ \psi_2 & \rightarrow & \psi_1. \end{array} \tag{4}$$

This also changes the orientation on the two-plane spanned by e_μ. (We note that the realization of $\mathbb{Z}_2$ is unique only up to phase factor.)

When we substitute the decomposition in the Yang–Mills action, the complex scalar fields $\psi_{1,2}$ become combined into the three-component unit vector [1]

$$\mathbf{t} = \frac{1}{\rho^2}(\psi_1^\star \quad \psi_2^\star)\,\vec{\sigma}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{1}{\rho^2}\begin{pmatrix}\psi_1^\star\psi_2 + \psi_2^\star\psi_1\\ \mathrm{i}(\psi_1\psi_2^\star - \psi_2\psi_1^\star)\\ \psi_1^\star\psi_1 - \psi_2^\star\psi_2\end{pmatrix} = \begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix}. \tag{5}$$

We have here parametrized

$$\psi_1 = \rho\,\mathrm{e}^{\mathrm{i}\zeta}\cos\tfrac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\phi/2}, \qquad \psi_2 = \rho\,\mathrm{e}^{\mathrm{i}\zeta}\sin\tfrac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi/2}. \tag{6}$$

The internal $U_I(1)$ transformation sends

$$t_\pm = \tfrac{1}{2}(t_1 \pm \mathrm{i}t_2) \rightarrow \mathrm{e}^{\mp 2\mathrm{i}\lambda}t_\pm \tag{7}$$

but t_3 remains intact. The $\mathbb{Z}_2$ action is a rotation that sends $(t_1, t_2, t_3) \rightarrow (t_1, -t_2, -t_3)$. In terms of the angular variables in (5) this corresponds to $(\phi,\theta) \rightarrow (2\pi-\phi, \pi-\theta)$. Thus we may opt to eliminate the $\mathbb{Z}_2$ degeneracy by a restriction to the upper hemisphere $\theta \in \left[0, \frac{\pi}{2}\right)$.

The off-diagonal components (1) determine the embedding of a two-dimensional plane in $\mathbb{R}^4$. The space of two-dimensional linear subspaces of $\mathbb{R}^4$ is the real Grassmannian manifold $Gr(4,2)$ [4]. It can be described by the antisymmetric tensor [1, 5, 6]

$$P_{\mu\nu} = \frac{\mathrm{i}}{2}\left(A_\mu^+ A_\nu^- - A_\nu^+ A_\mu^-\right) = A_\mu^1 A_\nu^2 - A_\nu^1 A_\mu^2 \tag{8}$$

that obeys the Plücker equation

$$P_{12}P_{34} - P_{13}P_{24} + P_{23}P_{14} = 0. \tag{9}$$

Conversely, *any* real antisymmetric matrix $P_{\mu\nu}$ that satisfies (9) can be represented in the functional form (8) in terms of two vectors A_μ^1 and A_μ^2. The Plücker equation describes the embedding of $Gr(4,2)$ in the five-dimensional projective space $\mathbb{RP}^5$ as a degree 4 hypersurface [4], a homogeneous space

$$Gr(4,2) \simeq \frac{SO(4)}{SO(2)\times SO(2)} \simeq \mathbb{S}^2 \times \mathbb{S}^2. \tag{10}$$

We note that similar geometric structures have been recently studied in the context of three-qubit entanglement [7] and in the context of stringy black hole solutions [8].

3. Internal $U_I(1)$ gauge symmetry

When we substitute (1) we get

$$P_{\mu\nu} = \frac{\mathrm{i}}{2}(|\psi_1|^2 - |\psi_2|^2)\cdot(e_\mu e_\nu^\star - e_\nu e_\mu^\star) = \frac{\mathrm{i}}{2}\cdot\rho^2\cdot t_3\cdot(e_\mu e_\nu^\star - e_\nu e_\mu^\star) = \rho^2\cdot t_3 H_{\mu\nu}. \tag{11}$$

This is clearly invariant under (3) and (4). In particular, we conclude that the vector field e_μ determines a $U_I(1)$ principal bundle over $Gr(4,2)$.

We employ $H_{\mu\nu}$ to explicitly resolve for the $U_I(1)$ structure as follows [1]. We first introduce the electric and magnetic components of (11),

$$E_i = \frac{\mathrm{i}}{2}(e_0 e_i^\star - e_i e_0^\star), \qquad B_i = \frac{\mathrm{i}}{2}\epsilon_{ijk} e_j^\star e_k. \tag{12}$$

They are subject to

$$\vec{E}\cdot\vec{B} = 0, \qquad \vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B} = \tfrac{1}{4}. \tag{13}$$

We then define the self-dual and anti-self-dual combinations

$$\vec{s}_\pm = 2(\vec{B} \pm \vec{E}). \tag{14}$$

This gives us two independent unit vectors that parametrize the 2-spheres $\mathbb{S}^2_\pm$ of our Grassmannian $Gr(4,2) \simeq \mathbb{S}^2_+ \times \mathbb{S}^2_-$, respectively. In these variables

$$\begin{aligned} e_\mu &= \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\eta}\cdot\left(\sqrt{1-\vec{s}_+\cdot\vec{s}_-},\ \frac{\vec{s}_+\times\vec{s}_- + \mathrm{i}(\vec{s}_- - \vec{s}_+)}{\sqrt{1-\vec{s}_+\cdot\vec{s}_-}}\right) \\ &= \mathrm{e}^{\mathrm{i}\eta}\cdot\left(\sqrt{2\vec{E}\cdot\vec{E}},\ \frac{2\vec{E}\times\vec{B} - \mathrm{i}\vec{E}}{\sqrt{2\vec{E}\cdot\vec{E}}}\right) \equiv \mathrm{e}^{\mathrm{i}\eta}\hat{e}_\mu. \end{aligned} \tag{15}$$

Here the phase factor η describes locally a section of the $U_I(1)$ bundle determined by e_μ over the Grassmannian (10). The $U_I(1)$ transformation sends $\eta \to \eta - \lambda$.

Since any two components of e_μ can vanish simultaneously, at least three coordinate patches for the base are needed in order to define the bundle. With local trivialization determined by $\eta_\alpha = \mathrm{Arg}(e_\alpha)$ these patches can be chosen to be $\mathcal{U}_\alpha = \{|e_\alpha| > \epsilon\}$ for $\alpha = 0, 1, 2$ with some (infinitesimal) $\epsilon > 0$. On the overlaps $\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta$ the transition functions are then

$$f_{\alpha\beta} = \exp\left\{\mathrm{i}\cdot \mathrm{Arg}\frac{e_\beta}{e_\alpha}\right\}$$

with e_α *resp.* e_β a component of vector e_μ that is nonvanishing in the overlap of $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$.

We now proceed to show by explicit computation the nontriviality of the $U_I(1)$ bundle. This implies that the phase factor η in (15) cannot be globally removed. We do this by relating our $U_I(1)$ bundle to the Dirac monopole bundle (Hopf fibration) $\mathbb{S}^3 \sim \mathbb{S}^2 \times \mathbb{S}^1$. We start by introducing the $U_I(1)$ connection

$$\Gamma = \mathrm{i}e^\star_\mu \mathrm{d}e_\mu = \mathrm{i}\hat{e}^\star_\mu \mathrm{d}\hat{e}_\mu + \mathrm{d}\eta = \hat{\Gamma} + \mathrm{d}\eta. \tag{16}$$

We locally parametrize the vectors $\vec{s}_\pm$ by

$$\vec{s}_\pm = \begin{pmatrix} \cos\phi_\pm \sin\theta_\pm \\ \sin\phi_\pm \sin\theta\pm \\ \cos\theta_\pm \end{pmatrix}. \tag{17}$$

We substitute this into (15) and (16). This gives us a (somewhat complicated) expression of $\hat{\Gamma}$ in terms of the angular variables (17). But when we compute the ensuing curvature 2-form the result is simple,

$$F = \mathrm{d}\hat{\Gamma} \equiv \mathrm{d}\Gamma = \sin\theta_+ \mathrm{d}\theta_+ \wedge \mathrm{d}\phi_+ + \sin\theta_- \mathrm{d}\theta_- \wedge \mathrm{d}\phi_-. \tag{18}$$

Consequently, the connection Γ in (16) is gauge equivalent to a connection of the form

$$\Gamma \sim -\cos\theta_+ \,\mathrm{d}\phi_+ - \cos\theta_- \,\mathrm{d}\phi_- + \mathrm{d}\eta. \tag{19}$$

When we restrict to one of the 2-spheres $\mathbb{S}_\pm$ in $Gr(4, 2)$ by fixing some point (pt) in the other, we obtain the two submanifolds $\mathbb{S}^2_+ \times$ pt and pt $\times\, \mathbb{S}^2_-$ and arrive at the functional form of the Dirac monopole connection in each of them. This establishes the fact that the $U_I(1)$ bundle is non-trivial and admits no global sections. In particular, the section η can only be defined *locally*.

We note that the appearance of the monopole line bundle can also be interpreted as follows: the two normalized orthogonal 4-vectors e^1 and e^2 form a real Stiefel manifold $V(4, 2) \simeq \mathbb{S}^3 \times \mathbb{S}^2$ (e.g. if the e^1_μ parametrizes $\mathbb{S}^3$, then e^2_μ which is perpendicular to e^1_μ parametrizes $\mathbb{S}^2$). When we account for the internal $U_I(1)$ this reduces $V(4, 2)$ to the real Grassmann manifold $V(4, 2)/U_I(1) \simeq \mathbb{S}^2 \times \mathbb{S}^2 \simeq Gr(4, 2)$ and the $U_I(1)$ fibration of $\mathbb{S}^3 \simeq \mathbb{S}^2 \times \mathbb{S}^1$ over $\mathbb{S}^2$ is the Hopf bundle.

4. Spacetime rotations and 1-cocycle

We now proceed to consider the linear action of (Euclidean) $SO(4)$ boosts. For this we rotate e_μ to a generic spatial direction ε_i ($i = 1, 2, 3$). In the case of an infinitesimal $\varepsilon = \sqrt{\vec{\varepsilon}\cdot\vec{\varepsilon}}$ the 4-vector e_μ ($\mu = 0, i$) transforms under the ensuing boost Λ_ε as follows:

$$\Lambda_\varepsilon e_0 = -\varepsilon_i e_i, \qquad \Lambda_\varepsilon e_i = -\varepsilon_i e_0. \tag{20}$$

For a finite ε the boost is obtained by exponentiation,

$$\begin{aligned} \mathrm{e}^{\Lambda_\varepsilon}(e_i) &= e_i + \frac{1}{\varepsilon^2}\cdot \varepsilon_i(\vec{e}\cdot\vec{\varepsilon}\cos(\varepsilon) + \varepsilon e_0 \sin(\varepsilon) - \vec{e}\cdot\vec{\varepsilon}), \\ \mathrm{e}^{\Lambda_\varepsilon}(e_0) &= e_0\cos(\varepsilon) - \frac{1}{\varepsilon}\vec{e}\cdot\vec{\varepsilon}\sin(\varepsilon) \equiv e_\mu \hat{\varepsilon}_\mu, \end{aligned} \tag{21}$$

where

$$0 \leqslant \varepsilon \equiv \sqrt{\vec{\varepsilon} \cdot \vec{\varepsilon}} < 2\pi \qquad (\text{mod } 2\pi)$$

and

$$\hat{\varepsilon}_\mu = \left(\cos(\varepsilon), -\sin(\varepsilon)\frac{\vec{\varepsilon}}{\varepsilon} \right).$$

We now identify a different, *projective* representation of $SO(4)$ on the Grassmannian: on the base manifold the ensuing $SO(4)$ boost acts on the electric and magnetic vectors $\vec{E}$ and $\vec{B}$ so that the result is the familiar

$$\Lambda_\varepsilon \vec{E} \equiv \delta_\varepsilon \vec{E} = \vec{B} \times \vec{\varepsilon}, \qquad \Lambda_\varepsilon \vec{B} \equiv \delta_\varepsilon \vec{B} = \vec{E} \times \vec{\varepsilon}. \tag{22}$$

For finite boost we get

$$e^{\delta_\varepsilon}(\vec{E}) = \frac{\vec{\varepsilon}(\vec{\varepsilon} \cdot \vec{E})(1 - \cos\varepsilon) + [\vec{B} \times \vec{\varepsilon}]\varepsilon \sin\varepsilon + \vec{E}\varepsilon^2 \cos\varepsilon}{\varepsilon^2} \tag{23}$$

and the same holds for the finite boost of $\vec{B}$, but with $\vec{E}$ and $\vec{B}$ interchanged.

We assert that the difference between (20) and (22), resp. (21) and (23), is a 1-cocycle, due to the projective nature of the second representation of $SO(4)$ on $Gr(4, 2)$. For this we recall the definition of a 1-cocycle: if ξ denotes a local coordinate system on $Gr(4, 2)$ and if a section of the $U_I(1)$ bundle which is locally specified by $e^{i\eta}$ is denoted by Ψ, then we have for a projective representation

$$\Lambda(g)\Psi(\xi) = \mathcal{C}(\xi, g)\Psi(\xi^g) \tag{24}$$

with $g \in SO(4)$. The factor $\mathcal{C}(\xi, g)$ is a 1-cocycle that determines the lifting of the projective representation to the linear representation. For a boost with the group element $g \in SO(4)$ which is parametrized by (finite) $\vec{\varepsilon}$ on the base manifold with $\vec{E}$ and $\vec{B}$, (24) becomes

$$e^{\Lambda_\varepsilon}\Psi(\vec{E}, \vec{B}) = \mathcal{C}(\vec{E}, \vec{B}, \vec{\varepsilon})\Psi(e^{\delta_\varepsilon}(\vec{E}), e^{\delta_\varepsilon}(\vec{B})). \tag{25}$$

We compute the 1-cocycle in (25) on a chart $\mathcal{U}_0$ with local trivialization $\eta = \text{Arg}(e_0)$. With $C(\xi, g) = \exp\{i\Theta(\xi, g)\}$ we look at the transformation of a local section $\exp\{\eta\}$ under the boost g. Under an infinitesimal boost the phase of e_0 changes as follows [1],

$$\Lambda_\varepsilon \eta = \Theta(\varepsilon) = \frac{\vec{E} \cdot \vec{\varepsilon}}{2\vec{E}^2} = \frac{(\vec{s}_+ - \vec{s}_-) \cdot \vec{\varepsilon}}{1 - \vec{s}_+ \cdot \vec{s}_-}. \tag{26}$$

For a finite boost we find from (26), by a direct exponentiation

$$\Theta(\vec{\varepsilon}) = \text{Arg}(\hat{e}_\mu \hat{\varepsilon}_\mu). \tag{27}$$

Note that this indeed reduces to (26) for infinitesimal ϵ. For general $g \in SO(4)$ we get in the chart $\mathcal{U}_0$

$$\Theta(\xi, g) = \text{Arg}\left(\frac{e_0^g}{e_0}\right). \tag{28}$$

Finally, since all one-dimensional representations are necessarily Abelian we conclude that Θ satisfies the 1-cocycle condition

$$\Lambda_{\varepsilon_1}\Theta(\vec{E}, \vec{B}; \vec{\varepsilon}_2) - \Lambda_{\varepsilon_2}\Theta(\vec{E}, \vec{B}; \vec{\varepsilon}_1) = 0$$

with its nontriviality following from the nontriviality of the Dirac monopole bundles.

5. Conclusions

In conclusion, we have established the nontriviality of the infinitesimal 1-cocycle found in [1] by relating it to the Dirac monopole bundle. We have also reported its finite version. The presence of the 1-cocycle establishes that in spin-charge separated Yang–Mills theory (Euclidean) boosts have two inequivalent representations, one acting linearly on the Grassmannian $Gr(2, 4)$ and the other projectively. The physical consequences of this observation remain to be clarified; in [1] a relation to Yang–Mills mass gap has been proposed.

Acknowledgments

We thank Ludvig Faddeev for discussions and comments. SS also thanks David Marsh for discussions. Our work has partially been supported by a VR Grant 2006-3376. The work by SS has also been partially supported by the Dmitri Zimin 'Dynasty' foundation, RSGSS-1124.2003.2; RFFI project grant 06-02-16786, by a STINT Institutional grant IG2004-2 025. AJN acknowledges the hospitality of CERN during the completion of this work.

References

[1] Faddeev L D and Niemi A J 2007 *Nucl. Phys.* B **776** 38
[2] Faddeev L D and Niemi A J 2002 *Phys. Lett.* B **525** 195
[3] Niemi A J and Walet N R 2005 *Phys. Rev.* D **72** 054007
[4] Hodge W V D and Pedoe D 1968 *Methods of Algebraic Geometry* vol I (Cambridge: Cambridge University Press)
[5] Marsh D 2007 *J. Phys. A: Math. Theor.* **40** 9919
[6] Slizovskiy S 2008 *J. Phys. A: Math. Theor.* **41** 065402
[7] Lévay P 2005 *Phys. Rev.* A **71** 012334
[8] Lévay P 2006 *Phys. Rev.* D **74** 024030